Hybrid Coordinate Dynamic Analysis," *AIAA Journal*, Vol. 6, No. 10, Oct. 1968, pp. 1867–1872.

[23] Hughes, P. C., "Flexibility Considerations for the Pitch Attitude Control of the Communications Technology Satellite," *CASI Transactions*, Vol. 5, No. 1, March 1972, pp. 1–4.

[24] Hughes, P. C., "Attitude Dynamics of a Three-Axis Stabilized Satellite With a Large Flexible Solar Array," *Journal of Astronautical Sciences*, to be published.

[25] Cherchas, D. B., "Attitude Stability and Performance of a Dual-Spin Satellite With Large Flexible Appendages," Rept. 170, Oct. 1971, Univ. of Toronto Institute for Aerospace Studies, Toronto, Ontario, Canada.

[26] Lamb, H., *Hydrodynamics*, 6th ed., Dover, New York, p. 622.

[27] Cloutier, G. J., "Nutation Damper Design Principles for Dual-Spin Spacecraft," *Journal of the Astronautical Sciences*, Vol. 16, No. 2, March–April 1969, pp. 79–87.

[28] Graham, W. B., "Linear Dynamics of Damped Cantilever Beams With Applications to a Symmetric Twin-Boom Satellite," M.A.Sc. thesis, 1971, Univ. of Toronto Institute for Aerospace Studies, Toronto, Ontario, Canada.

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Orbit Determination by Range-Only Data

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The determination of satellite orbits for use in geodesy using range-only data has been examined. A recently developed recursive algorithm for rectification of the nominal orbit after processing each observation has been tested. It is shown that when a synchronous satellite is tracked simultaneously with a subsynchronous geodetic target satellite, the orbits of each may be readily determined by processing the range information. Random data errors and satellite perturbations are included in the examples presented.

Nomenclature

a = semimajor axis of satellite orbit, Earth radii
e = eccentricity of satellite orbit, nondimensional
i = inclination of satellite orbit plane, radians
Ω = longitude of the ascending node of satellite orbit, radians
ω = argument of perigee of satellite orbit, radians
T = epoch of perigee passage of satellite, seconds
E = eccentric anomaly of satellite, radians
ρ_o = observed range from tracker to target, Earth radii
ρ_c = computed range from tracker to target, Earth radii
$\delta\rho$ = range residual, Earth radii
V_x, V_y, V_z = satellite velocity components, Earth radii/sec
X, Y, Z = satellite coordinates, Earth radii
v_i = observation error in the ith measurement
d = distance from tracker to geocenter, Earth radii
r = distance from target satellite to geocenter, Earth radii
m = set of orbital parameters

Subscripts

T = target satellite
S = tracking satellite
$*$ = nominal trajectory

Introduction

RECENT work in satellite geodesy has demonstrated the feasibility of including the differential equations describing the rotational motion of a deformable earth in an orbit-determination model. By including the Earth dynamical relationships it is possible to process range observations to determine the motion of the pole, the Love numbers h and l, and perhaps to predict the occurrence of major earthquakes. Any parameter which has a direct geometrical effect on satellite range data might be determined in this manner. Therefore it is of interest to consider the determination of geodetic parameters simultaneously with the determination of orbits using range-only data. This paper presents the results of the investigations of techniques for orbit determination using range-only data.

Techniques involving range-only data for orbit determination have been investigated by several researchers in the past. The method of Baker,[1] based on the synthesis of the classical f and g series of celestial mechanics, has been applied to orbit determination using range-only data obtained from ground-based tracking stations. The possibility of using range-only data to determine circular orbits has been also demonstrated by Ball.[2] This technique was dependent on specific geometric properties existing between the tracker and the target satellite's orbit. The feasibility of using two geostationary tracking satellites, called Ephemeris Determination Satellites (EDS), to track target satellites in low-altitude polar and inclined orbits has also been investigated.[3] The accuracies obtained with an orbit-determination computer program using a minimum-variance estimation process and with the use of both range and range-rate measurements from EDS to target satellite were shown to be comparable with those of ground tracking networks.

Space-based tracking stations have the following advantages over ground-based tracking: 1) For a synchronous orbital tracking station the tracking function can be combined with synchronous communication relays. 2) Tracking coverage from space-based stations is increased greatly over that obtainable from an Earth-based station. 3) Because some range measurements can be made outside the atmosphere, the errors that result from atmospheric distortion and refraction are eliminated. 4) Space-based tracking stations can give better geometry between stations and target satellites, therefore providing better accuracy.

Description of the Method

In the following development it has been assumed that both tracking and target satellites are on elliptical orbits. This will not affect the generality of the proposed approach since one can use simple transformations from the expressions developed below to get desired results for the circular or hyperbolic cases.

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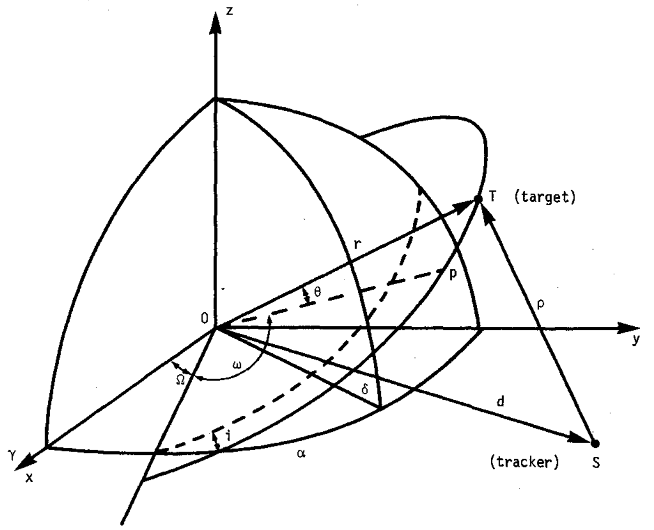

Fig. 1 Positions of tracking and target satellites.

From the basic vector relationship (see Fig. 1)

$$\boldsymbol{\rho} = \mathbf{r} - \mathbf{d}$$

one can write

$$\boldsymbol{\rho} \cdot \boldsymbol{\rho} = \rho^2 = r^2 + d^2 - 2\mathbf{r} \cdot \mathbf{d} \tag{1}$$

Using the geocentric equatorial coordinate frame $0xyz$ the coordinates of any Earth satellite may be expressed in terms of its orbital elements through the use of the eccentric anomaly. Thus, one has

$$\begin{aligned} X &= a\{(\cos E - e)(\cos\Omega \cos\omega - \sin\Omega \sin\omega \cos i) - (1 - e^2)^{1/2} \\ &\quad \sin E (\cos\Omega \sin\omega + \sin\Omega \cos\omega \cos i)\} \\ Y &= a\{(\cos E - e)(\sin\Omega \cos\omega + \cos\Omega \sin\omega \cos i) - (1 - e^2)^{1/2} \\ &\quad \sin E (\sin\Omega \sin\omega - \cos\Omega \cos\omega \cos i)\} \\ Z &= a \sin i\{(\cos E - e) \sin\omega + (1 - e^2)^{1/2} \sin E \cos\omega\} \end{aligned} \tag{2}$$

Equation 1 then becomes

$$\rho^2 = a_s^2(1 - e_s \cos E_s)^2 + a_T^2(1 - e_T \cos E_T)^2 - 2(X_S X_T + Y_S Y_T + Z_S Z_T) \tag{3}$$

where X_S, Y_S, Z_S, X_T, Y_T, Z_T are coordinates of tracking and target satellites, respectively, given by Eq. (2) with the corresponding set of orbital parameters.

From Eq. (3) one can write the expression for ρ in the form

$$\rho = \rho(a_s, e_s, i_s, \Omega_s, \omega_s, E_s, a_T, e_T, i_T, \Omega_T, \omega_T, E_T).$$

Let ρ_0 be the observed range from tracker to target satellite and ρ_c the computed range from Eq. (3), based on the first estimates of position and velocity vectors of the satellites, both tracker and target, at a certain epoch $t = t_1$. With Kepler's equation

$$E - e \sin E = (\mu/a^3)^{1/2}(t - T)$$

one can express the range residuals $\delta\rho = \rho_0 - \rho_c$ in terms of the corrections

$$\delta a_s, \delta e_s, \delta i_s, \delta\Omega_s, \delta\omega_s, \delta T_s, \delta a_T, \delta e_T, \delta i_T, \delta\Omega_T, \delta\omega_T, \,\&\, \delta T_T$$

as follows

$$\begin{aligned} \delta\rho = {} & \{(\partial\rho/\partial a_s) + (\partial\rho/\partial E_s) \cdot (\partial E_s/\partial a_s)\}_* \delta a_s + \\ & \{(\partial\rho/\partial e_s) + (\partial\rho/\partial E_s) \cdot (\partial E_s/\partial e_s)\}_* \delta e_s + \{(\partial\rho/\partial i_s)\}_* \delta i_s + \\ & \{(\partial\rho/\partial\Omega_s)\}_* \delta\Omega_s + \{(\partial\rho/\partial\omega_s)\}_* \delta\omega_s + \{(\partial\rho/\partial E_s) \cdot \\ & (\partial E_s/\partial T_s)\}_* \delta T_s + \{(\partial\rho/\partial a_T) + (\partial\rho/\partial E_T) \cdot (\partial E_T/\partial a_T)\}_* \delta a_T + \\ & \{(\partial\rho/\partial e_T) + (\partial\rho/\partial E_T) \cdot (\partial E_T/\partial e_T)\}_* \delta e_T + \\ & \{(\partial\rho/\partial i_T)\}_* \delta i_T + \{(\partial\rho/\partial\Omega_T)\}_* \delta\Omega_T + \{(\partial\rho/\partial\omega_T)\}_* \delta\omega_T + \\ & \{(\partial\rho/\partial E_T) \cdot (\partial E_T/\partial T_T)\}_* \delta T_T \end{aligned} \tag{4}$$

where the partial derivatives are evaluated on nominal trajectories. For unperturbed motions, from the set of equations (4) (at least 12, one for each measurement of range) one can solve for the best corrections to the previous estimates of the orbital parameters by the method of least-squares. The new values of the orbital elements are substituted back into Eqs. (2) and (3) to get the new values for ρ_c, and the new range residuals are computed again. This process is repeated until the range residuals are sufficiently small.

In the general case of perturbed motions the differential equations for the orbital element m can be expanded in a Taylor series about the nominal m^* to get, for a first order approximation, $\delta\dot{m} = F\delta m$ where F is the matrix containing the partial derivatives of $\dot{m}$ with respect to m, evaluated at m_*. The solution for the perturbations may be determined in terms of the state transition matrix in the usual manner. One has

$$\delta m_k = \Phi(t_k, t_j)\delta m_j \tag{5}$$

Let v_i be the error in the ith range measurement. v_i is assumed to be sampled from a gaussian sequence having mean zero and unknown but fixed covariance matrix R. Equation (4) can be written as

$$\delta\rho_i = \tilde{H}_i \delta m_i + v_i, \qquad i = 1, \ldots, N \tag{6}$$

where N is the number of observations.

Let

$$H_i = \tilde{H}_i \Phi(t_i, t_k)$$

one then has

$$\delta\rho_i = H_i \delta m_k + v_i \tag{7}$$

and the determination of optimal changes of orbital parameters of tracking and target satellites from the nominal values can be put in the following framework of an estimation problem.

Given a set of range residuals, $\delta\rho_i$, $i = 1, \ldots, N$, the plant equation (5) for the perturbed state, the observation model (7), and the a priori statistics $E[\delta m_0] = 0$, $E[\delta m_0 \delta m_0^T] = P_0$, find a sequential estimation to get the best estimate of δm_k and the observation-error covariance matrix R at time t_k.

The recent algorithm suggested by Tapley and Born[4] is suitable for this type of problem.

Preliminary Results

The above formulation has been applied to determine the orbital parameters of an Earth satellite on an elliptical inclined orbit for three specific cases: 1) tracking the target satellite by a ground station; 2) tracking the target satellite by another satellite on a known perturbed orbit; 3) using a ground station to track the "tracking satellite" and, at the same time, using the latter to track the target satellite.

The coordinates of the ground station and the initial orbital elements of nominal orbits of both tracking and target satellites are given in Table 1.

The only perturbation introduced in the examples is that due to the Earth's oblateness. The potential field for the nonspherical Earth can be represented to various levels of accuracy

Table 1 Coordinates of ground station and orbital parameters

Ground station	Tracking satellite	Target satellite
$\alpha = 2.426788$ (rad)	$a_s = 6.592185$ (radii)	$a_T = 1.6$ (radii)
	$e_s = 0.05$	$e_T = 0.1$
$\delta = 0.5934$ (rad)	$i_s = 0.02$ (rad)	$i_T = 1.138991$ (rad)
	$\Omega_s = 1.5$ (rad)	$\Omega_T = 1.796065$ (rad)
	$\omega_s = 0.7$ (rad)	$\omega_T = 1.004865$ (rad)
	$T_s = T_T + 1000$ (sec)	$T_T = 0$ (set arbitrarily)

Table 2 Results from re-estimation of the initial state—case (1)

	a	e	i	Ω	ω	$T \times 10^3$	Range
Actual	1.5813	0.12534	1.1329	1.8110	1.0163	2.1829	2.1936
Initial estimate	1.6000	0.10000	1.1389	1.7960	1.0048	0	2.2275
1	1.5779	0.12769	1.1368	1.7989	1.0065	2.1131	2.1904
2	1.5791	0.12655	1.1366	1.7982	1.0053	1.9797	2.1934
3	1.5791	0.12659	1.1366	1.7982	1.0052	1.9707	2.1934
Error	-2.2×10^{-3}	1.25×10^{-3}	3.7×10^{-3}	-1.28×10^{-2}	-1.11×10^{-2}	2.122×10^{-4}	2.0×10^{-4}

Table 3 Results from re-estimation of the initial state—case (2)

	a	e	i	Ω	ω	$T \times 10^3$	Range
Actual	1.6014	0.09311	1.1471	1.8064	1.0145	3.9829	7.1787
Initial estimate	1.6000	0.1000	1.1389	1.7960	1.0048	0	7.5494
1	1.5905	0.085707	1.1419	1.7949	1.0060	4.0080	7.1487
2	1.5872	0.099735	1.1426	1.7938	1.0066	4.0376	7.1636
3	1.5856	0.097953	1.1431	1.7938	1.0069	4.0531	7.1684
Error	-1.58×10^{-2}	4.843×10^{-3}	-4.0×10^{-3}	-1.26×10^{-2}	-7.6×10^{-3}	7.05×10^{-5}	-1.03×10^{-2}

Table 4 Results from re-estimation of the initial state—case (3)

	a	e	i	Ω	ω	$T \times 10^3$	Range
Actual	1.6014	0.09311	1.1471	1.8064	1.0145	3.9829	7.1787
Initial estimate	1.6000	0.10000	1.1389	1.7960	1.0048	0	7.5494
1	1.6015	0.081002	1.1390	1.7915	1.0045	3.6063	7.1544
2	1.5948	0.093016	1.1407	1.7954	1.0056	3.7452	7.1801
3	1.5949	0.093269	1.1407	1.7957	1.0056	3.7477	7.1802
Error	6.5×10^{-3}	1.59×10^{-4}	6.4×10^{-3}	1.07×10^{-2}	8.9×10^{-3}	2.352×10^{-4}	1.5×10^{-3}

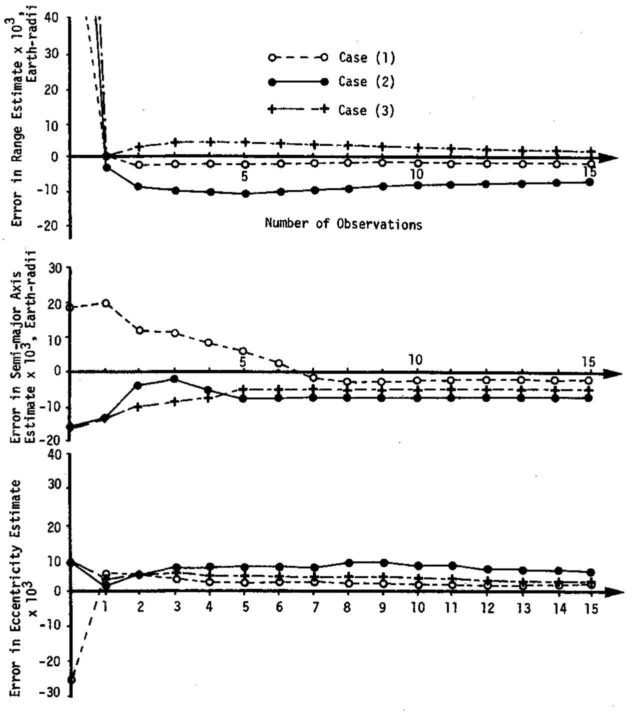

Fig. 2 Comparison of range, semimajor axis, and eccentricity estimates.

by a series of spherical harmonics. For the purpose of this study only first-order perturbations were used with

$$R_1 = \tfrac{3}{2}\mu(J/a^3)(\tfrac{1}{3}-\tfrac{1}{2}\sin^2 i)(1-e^2)^{-3/2}$$

$$\mu = 3.98602 \times 10^5 \text{ km}^3/\text{sec}^2, \qquad J_2 = 1082.645 \times 10^{-6}$$

The perturbation equations, derived from Lagrange equations, are given in Ref. 5.

The observed ranges were simulated by a program written in double precision for Fortran IV on a CDC 6400 digital computer. The entire orbit determination program was also written in double precision because of the number of trigonometric functions involved in the various expressions and to avoid the detrimental effect of truncation which may cause the error variance matrix to lose its positive definiteness or symmetry and hence lead to divergence of the filter used in the estimation of orbital parameters.

The program started by assuming $\delta m_0 = 0$. Using 15 observations with a 60-sec interval between two observations (and caution has been taken to enforce the symmetry of the error-variance matrix at each step), the sequential estimation algorithm developed in Ref. 4 converged to 4 decimal places for case 1 and 3 decimal places for cases 2 and 3 after 3 iterations. The results are shown in Tables 2, 3, and 4. The comparisons of the estimates of ranges and orbital parameters of the target satellite in the three cases for various numbers of observations are given in Figs. 2–4. A modified version of the Tapley-Born

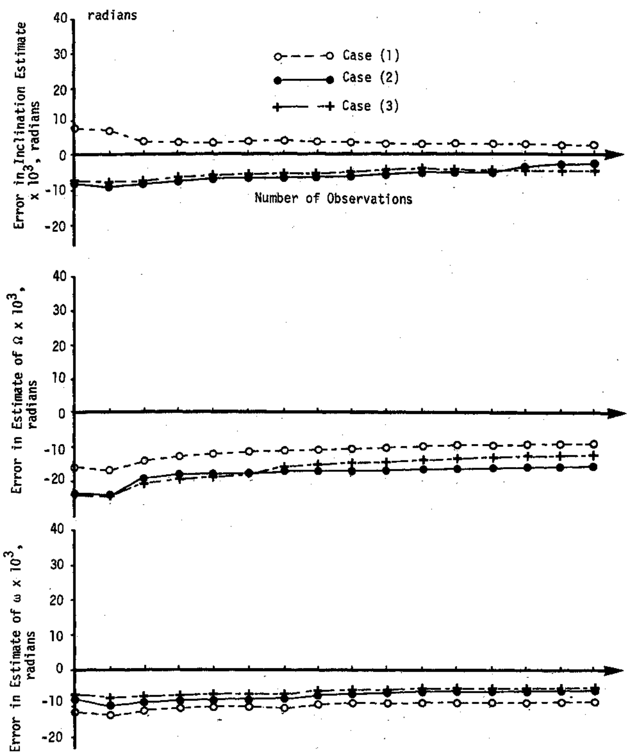

Fig. 3 Comparison of inclination, ascending node, and argument of perigee estimates.

algorithm used to rectify the nominal orbit after processing each observation has also been tested. The improvements in the estimates of range and orbital parameters of the target satellite for case 1 are shown in Figs. 5 and 6.

The following results can be drawn: 1) Since the effect of atmospheric distortion was neglected, the use of a ground station to track a target satellite always gave the least estimation error because of the known coordinates of the tracker. 2) Case 3 gave better results than case 2 because the tracking of the tracking satellite gave one additional "constraint" to the estimation problem for the target satellite. 3) All three cases show acceptable estimation-error after very few iterations. 4) The modified version of the Tapley-Born sequential estimation algorithm based on orbit rectification at each observation gave better results with a slight increase in computer time.

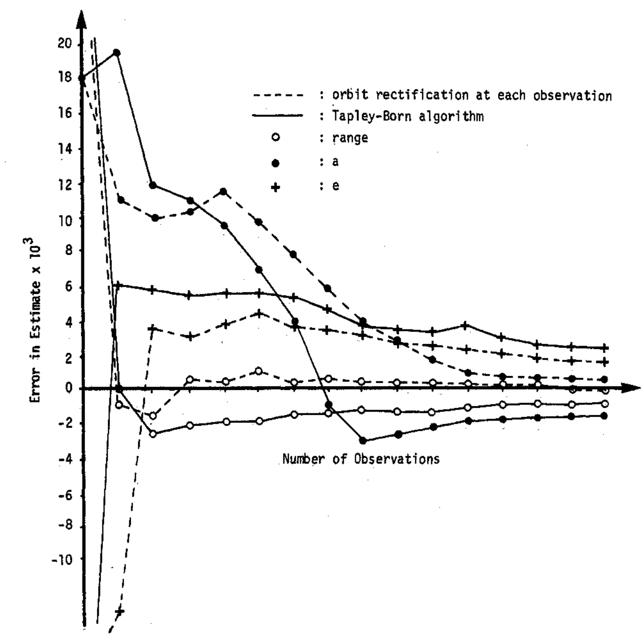

Fig. 5 Improvement in the estimates of ρ_c, a, e by rectifying nominal orbit at each observation.

Conclusions

Within the next few years, it is expected that advances in measurement technology will permit the resolution of the Earth's rotational and deformational motion to the level of 2 cm–1 m. With this in mind, a discrete time Kalman filtering algorithm has been developed for processing any set of geodetic observations which directly involves the angular or deformational motion of the Earth.[6] Important applications of this Earth dynamic filter are the measurement of variations in the Earth's rotation rate, the movement of the pole, continental drift, the determination of the Love numbers h and l, and possible earthquake prediction.

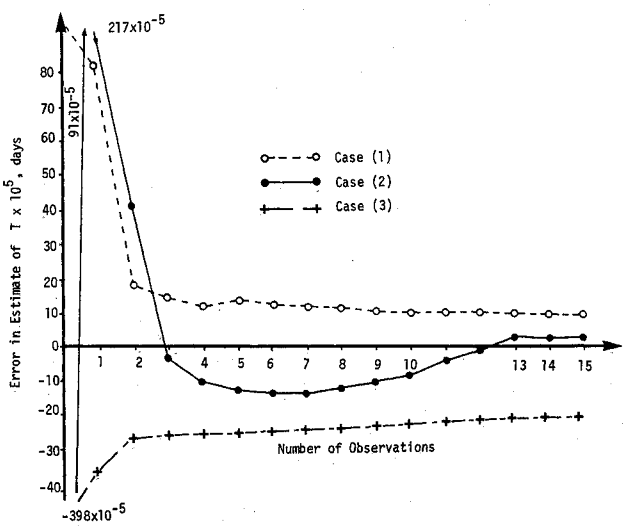

Fig. 4 Comparison of epoch of perigee passage estimates.

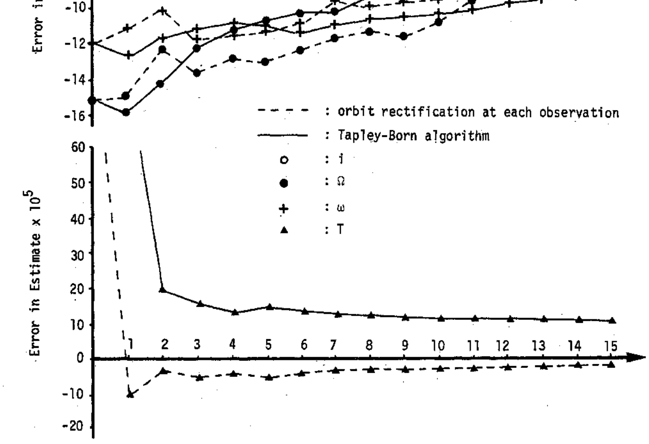

Fig. 6 Improvement in the estimates of i, Ω, ω, and T by rectifying nominal orbit at each observation.

Numerical experiments have been conducted involving simulated range measurements between a proposed geodetic satellite (the Cannonball satellite[7]) and a global network of tracking stations.[6] The object was to determine to what degree parameters having a direct geometrical effect on satellite range data could be estimated. It was determined that, with a noise level of 2 cm in range observations, the range observations contain recoverable information about the deformation properties of the Earth and that the change in position of the secular pole due to a large scale mass redistribution (such as that which presumably precedes a major earthquake[8]) may be obtained from the observations. However, it was also found that the use of the single satellite did not provide adequate information to determine the coordinates of the rotation pole to any higher accuracy than that which is presently obtained by the International Polar Motion Service, the Bureau International de l'Heure, and the U.S. Naval Weapons Laboratory.[9] The use of more than one geodetic satellite would provide sufficient geometric information to be able to extract the coordinates of the pole position.

Since many of the problems of current interest in geodesy involve the relationships between geodetic parameters and their geometrical effects on satellite range data, the use of range data for both orbit determination and geodesy appears logical. In some cases, however, it may turn out that for a given satellite constellation the state vector cannot be augmented by certain geodetic parameters due to the difficulty of separating range partials. Also, many geodetic phenomena of interest (such as continental drift) occur on time scales that do not require real-time data processing. For those cases the filter should be replaced by a fixed-delay smoothing operation.

The theoretical developments and numerical examples of the orbit determination method using a single space-based tracking station providing range-only data have demonstrated that this method works well, especially when the co-ordinates of the tracking satellite are known with sufficient accuracy or when one conducts the tracking of the tracking satellite simultaneously with the tracking of the target satellite by the tracking satellite. In addition, the derivations of the expressions for range in this method do not depend on specific geometric properties existing between the tracker and target satellites; hence this is a general method for orbit determination using a single space-based tracking station and range-only data. The accuracy in the estimation of orbital elements of the target satellite depends very much on the approximation of roots of Kepler's equation and the integration step-size used to evaluate the state-transition matrix.

References

[1] Baker, R. M. L., Jr., "Preliminary Results Concerning Range-only Orbit Determination," *Proceedings of the 1st International Symposium on Analytical Astrodynamics*, 1961, Univ. of California, Los Angeles, p. 61.

[2] Ball, J. E., "Preliminary Circular Orbit from a Single Station of Range-only Data," *AIAA Journal*, Vol. 4, No. 12, Dec. 1967, pp. 2264–2265.

[3] Johnson, D. J., Mullin, F. J., and Steiner, P., "Synchronous Satellites as Part of a Complex Tracking Network," *Journal of Spacecraft and Rockets*, Vol. 6, No. 6, June 1969, pp. 679–684.

[4] Tapley, B. D. and Born, G. H., "Sequential Estimation of the State and the Observation-Error Covariance Matrix," *AIAA Journal*, Vol. 9, No. 2, Feb. 1971, pp. 212–217.

[5] McCuskey, S. W., *Introduction to Celestial Mechanics*, Addison-Wesley, Reading, Mass., 1963, p. 143.

[6] McClure, P., "Earth Dynamic Filtering for Earthquake Prediction and Love Number Determination," Rept. ME 72-1, March 1972, Dept. of Mechanical Engineering, Colorado State Univ., Fort Collins, Colo.

[7] Weiffenbach, G. and Hoffman, T., "A Passive Stable Satellite for Earth-Physics Applications," Smithsonian Astrophysical Observations Special Report 329, 1970, Smithsonian Inst., Cambridge, Mass.

[8] Smylie, D. and Mansinha, L., "Earthquakes and the Observed Motion of the Rotation Pole," *Journal of Geophysical Research*, Vol. 73, No. 24, Dec. 1968, pp. 7661–7673.

[9] Anderle, R., "Polar Motion Determinations by U.S. Navy Doppler Satellite Observations," NWL Technical Report, TR-2432, July 1970, Naval Weapons Lab., Dahlgren, Va.